

ON THE ZEROS OF WEAKLY HOLOMORPHIC MODULAR FORMS

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ABSTRACT. In this article, we study the nature of zeros of weakly holomorphic modular forms. In particular, we prove results about transcendental zeros of modular forms of higher levels and for certain Fricke groups which extend a work of Kohnen (see [14]). Furthermore, we investigate the algebraic independence of values of weakly holomorphic modular forms.

1. INTRODUCTION

Throughout the paper, let $\mathfrak{H} := \{z \mid \Im(z) > 0\}$ be the upper half-plane, k be an even natural number and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ be the full modular group. Further, a CM point is an element of \mathfrak{H} lying in an imaginary quadratic field. A holomorphic function f on \mathfrak{H} is called a weakly holomorphic modular form of weight k for Γ if for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } z \in \mathfrak{H}$$

and f has an expansion of the form

$$(1) \quad f(z) = \sum_{n \geq n_0} a(n) e^{2\pi i n z}, \text{ where } z \in \mathfrak{H}.$$

Moreover, if $n_0 = 0$ (resp. $n_0 = 1$) in equation (1), then we call f a modular form (resp. cusp form) of weight k for Γ . From now on, we will denote weakly holomorphic modular forms by WH modular forms. Also denote $e^{2\pi i z}$ by q where $z \in \mathfrak{H}$. Some well known examples of WH modular forms are Eisenstein series of weight $k \geq 4$ defined by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$, B_k 's are Bernoulli numbers, the normalized weight 12 Delta function

$$\Delta(z) := q \prod_{n \geq 1} (1 - q^n)^{24}$$

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and the weight zero j -function given by

$$j(z) := 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2}.$$

For other examples of families of WH modular forms, see [1, 7, 11].

In this article, we study the nature of zeros of WH modular forms. From now on, by zeros of f , we mean inequivalent zeros of f , i.e. we investigate zeros of f in \mathfrak{F} , where

$$\mathfrak{F} := \{z \in \mathfrak{H} \mid |z| \geq 1, -1/2 \leq \Re(z) \leq 0\} \cup \{z \in \mathfrak{H} \mid |z| > 1, 0 < \Re(z) < 1/2\}$$

is the standard fundamental domain. We will restrict our attention to forms whose Fourier coefficients $a(n)$ in equation (1) are in a subfield F of $\overline{\mathbb{Q}}$. These WH modular forms (resp. modular forms and cusp forms) of weight k for Γ form a vector space over F denoted by $M_k^!(F)$ (resp. by $M_k(F)$ and $S_k(F)$).

The paper consists of theorems of three different types. First we discuss about the existence of at least one transcendental zero of certain WH modular forms and use this information to provide a criterion to study the nature of Fourier coefficients of WH modular forms.

Next we show that most of the zeros of certain families of WH modular forms for level one and higher level and for certain Fricke groups (defined later) are transcendental. In particular, our results give a generalization of a theorem of Kohnen [14]. An essential ingredient in the proof of these theorems is the *q-expansion principle* due to Deligne and Rapoport [6] (see Theorem 3.9, p. 304).

In the final set of theorems, we study the values of WH modular forms at certain algebraic and transcendental numbers.

The motivation to prove these results come from the classical results of Chowla-Selberg [2], Chudnovsky [3, 4], Nesterenko [17], Ramachandra [19] and Schneider [22] and the recent works of the first author with Murty and Rath [10], Kanou [12] and Kohnen [14].

2. STATEMENT OF THEOREMS

In this section, we list the theorems we prove in the final section.

Theorem 1. *Let $f(z) = \sum_{n \geq n_0} a(n)q^n \in M_k^!(\overline{\mathbb{Q}})$ with $a(n_0) \in \overline{\mathbb{Z}} \setminus \{0\}$ and not all $a(n) \in \overline{\mathbb{Z}}$. Then f has at least one transcendental zero.*

Remark 2.1. Let $\beta \in \mathfrak{H}$ be a transcendental number such that $j(\beta) \in \overline{\mathbb{Q}} \setminus \overline{\mathbb{Z}}$. Examples of infinitely many such transcendental numbers are given in [9]. For $z \in \mathfrak{H}$, define a weakly holomorphic modular form f of weight k such that

$$f(z) = \Delta^{k/12}(z) (j(z) - j(\alpha_1)) \cdots (j(z) - j(\alpha_n)) (j(z) - j(\beta)),$$

where $\alpha_1, \dots, \alpha_n \in \mathfrak{H}$ are CM points. Then f satisfies the assumptions of Theorem 1. Note that f has exactly one transcendental zero and hence Theorem 1 is the best possible.

Remark 2.2. Below we construct examples to show that both the hypothesis in Theorem 1 are necessary. Given a finite set of CM points $\alpha_1, \dots, \alpha_n$ and $z \in \mathfrak{H}$, we can define a weakly holomorphic modular form f of weight k such that

$$(2) \quad f(z) = c \Delta^{k/12}(z) (j(z) - j(\alpha_1)) \cdots (j(z) - j(\alpha_n)), \text{ where } c \in \overline{\mathbb{Q}} \setminus \overline{\mathbb{Z}}.$$

Then f has Fourier coefficients in $\overline{\mathbb{Q}} \setminus \overline{\mathbb{Z}}$ and only algebraic zeros. By taking $g(z) := f(z)/c$, where f is as in equation 2, we see that g has all coefficients in $\overline{\mathbb{Z}}$ but has only algebraic zeros.

Define

$$M_k^!(\overline{\mathbb{Z}}) := \left\{ f(z) = \sum_{n \geq n_0} a(n)q^n \in M_k^!(\overline{\mathbb{Q}}) \mid a(n) \in \overline{\mathbb{Z}} \forall n \right\}.$$

Using Theorem 1, we give a criterion for an $f \in M_k^!(\overline{\mathbb{Q}})$ to have algebraic integer Fourier coefficients.

Corollary 2. Suppose that $f(z) = \sum_{n \geq n_0} a(n)q^n$ is a WH modular form of weight k , where $a(n_0)$ is a non-zero algebraic integer. Further suppose that all zeros of f are algebraic. Then $f \in M_k^!(\overline{\mathbb{Q}})$ if and only if $f \in M_k^!(\overline{\mathbb{Z}})$.

Our next set of theorems show that information about location of zeros can be useful to determine the nature of zeros. These theorems can be thought of as generalizations of a theorem of Kohnen (see [14]).

Theorem 3. Let $f \in M_k^!(\mathbb{Q})$ be a weakly holomorphic modular form with all its zeros on A , where

$$A := \{e^{i\theta} \mid \pi/2 \leq \theta \leq 2\pi/3\}.$$

Then all zeros of f other than the possible zeros at i and $\rho = e^{2i\pi/3}$ are transcendental.

Remark 2.3. Rankin and Swinnerton-Dyer [20] showed that the Eisenstein series E_k for even $k \geq 4$ have all their zeros on the arc A . See [1, 7, 11] for other examples of WH modular forms having similar properties. Contrary to these forms, zeros of Hecke eigen cusp forms are equidistributed in \mathfrak{F} with respect to the hyperbolic measure (see [21] and [24] for further details).

Using Theorem 3, we now prove a higher level analogue. For a natural number N , let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}.$$

A holomorphic function f on \mathfrak{H} is called a WH modular form of weight k and level N if for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } z \in \mathfrak{H}$$

and f is meromorphic at all its cusps. Further, if f is holomorphic at all cusps, we call f a modular form. We will consider inequivalent zeros of such WH modular forms in the fundamental domain

$$\mathfrak{F}^{(N)} := \cup_{\gamma \in \Gamma/\Gamma_0(N)} \gamma \mathfrak{F}.$$

In this set-up, we prove the following theorem.

Theorem 4. *Let p be a prime and f be a WH modular form of weight k and level p having rational Fourier coefficients at the cusp $i\infty$. Suppose that all zeros of f lie on*

$$B := \cup_{\gamma \in \Gamma/\Gamma_0(p)} \gamma A,$$

where A is as in Theorem 3. Let

$$C := \left\{ \frac{i-n}{n^2+1} \mid 0 \leq n < p \right\} \cup \left\{ \frac{\frac{\sqrt{3}}{2}i-n+\frac{1}{2}}{n^2-n+1} \mid 0 \leq n < p \right\}.$$

Then all zeros of f except the possible CM zeros lying in C are transcendental.

Next we study transcendental zeros of Eisenstein series for certain Fricke groups $\Gamma_0^*(p)$, where

$$\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)W_p, \quad W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$$

and p is a prime. Here we will study zeros in the fundamental domain

$$\mathfrak{F}_p := \{|z| \geq 1/\sqrt{p}, -1/2 \leq \Re(z) \leq 0\} \cup \{|z| > 1/\sqrt{p}, 0 < \Re(z) < 1/2\}.$$

One can define modular forms on Fricke groups analogously to modular forms for congruence subgroups. The Eisenstein series for $\Gamma_0^*(p)$ is defined by

$$E_{k,p}(z) := \frac{1}{p^{k/2}+1} \left(p^{k/2} E_k(pz) + E_k(z) \right),$$

where $E_k(z)$ is the Eisenstein series for Γ . For $p = 2, 3$, the location of zeros of $E_{k,p}(z)$ were studied by Miezaki, Nozaki and Shigezumi (see [16]). They showed that the zeros of $E_{k,p}(z)$ lie on the arc

$$A_p := \{z \in \mathfrak{H} \mid -1/2 \leq \Re(z) \leq 0, |z| = 1/\sqrt{p}\}.$$

The nature of the zeros of the Eisenstein series E_k for the full modular group Γ was first studied by Kanou [12]. He showed that E_k for even $k \geq 16$ has at least one transcendental zero in \mathfrak{F} . Soon after, Kohnen [14] proved that any zero of E_k in \mathfrak{F} different from i or ρ is necessarily transcendental. See also [9] for similar results. Here we prove:

Theorem 5. *Let $E_{k,2}(z)$ be the Eisenstein series of weight k for the Fricke group $\Gamma_0^*(2)$. Then all zeros of $E_{k,2}(z)$ other than the possible zeros at*

$$\frac{i}{\sqrt{2}}, \quad \frac{-1+i\sqrt{7}}{4}, \quad \frac{-1+i}{2}$$

are transcendental.

Analogously, we have the following theorem for $E_{k,3}(z)$.

Theorem 6. *Let $E_{k,3}(z)$ be the Eisenstein series of weight k for the Fricke group $\Gamma_0^*(3)$. All zeros of $E_{k,3}(z)$ are transcendental other than the following possible CM zeros*

$$\frac{i}{\sqrt{3}}, \quad \frac{-1 + i\sqrt{11}}{6}, \quad \frac{-1 + i\sqrt{2}}{3}, \quad \frac{-3 + i\sqrt{3}}{6}.$$

The notion of real zeros of modular forms was introduced by Ghosh and Sarnak in [8] and can be extended to WH modular forms. A zero z_0 of a WH modular form f is called real if it lies on the arc A or it lies on the vertical line passing through ρ or the vertical line passing through i . In this context, we have the following theorems.

Theorem 7. *Let $f \in M_k^!(\mathbb{Q})$ be a weakly holomorphic modular form with all its zeros on L , where*

$$L := -\frac{1}{2} + it \quad \text{with } t \geq \frac{\sqrt{3}}{2}.$$

Then any CM zero of f (if exists) is necessarily of the form

$$-\frac{1}{2} + i\frac{\sqrt{|D|}}{2a}, \quad \text{where } 1 \leq a \leq \left\lceil \sqrt{\frac{|D|}{3}} \right\rceil, \quad a \in \mathbb{N}.$$

Here D is a discriminant of an imaginary quadratic field which is necessarily congruent to 1 (mod 4).

Remark 2.4. Let f be a WH modular form as in Theorem 7. Note that the above theorem helps us to calculate the possible CM zeros with bounded imaginary parts of such a form. In particular if we consider the collection

$$M_L := \{f \in M_k^!(\mathbb{Q}) \mid f(\alpha) = 0 \implies \alpha \in L\},$$

then all zeros of $f \in M_L$ on the line segment

$$L_2 := \{z \in L \mid \Im(z) < 2\}$$

are transcendental except for

$$\rho, \quad -\frac{1}{2} + i\frac{\sqrt{7}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{11}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{15}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{15}}{4}.$$

Theorem 8. Let $f \in M_k^1(\mathbb{Q})$ be a WH modular form with all its zeros on

$$R := it \quad \text{with } t \geq 1.$$

Then any CM zero of f (if exists) is of the form

$$i \frac{\sqrt{|D|}}{2a} \quad \text{with } 1 \leq a \leq \left\lfloor \frac{\sqrt{|D|}}{2} \right\rfloor, \quad a \in \mathbb{N}.$$

Here D is a discriminant of an imaginary quadratic field and necessarily $D \equiv 0 \pmod{4}$.

Remark 2.5. As before, by applying Theorem 8 we can calculate possible CM zeros with bounded imaginary part for forms of the above type. In particular, if we consider the collection

$$M_R := \{f \in M_k^1(\mathbb{Q}) \mid f(\alpha) = 0 \implies \alpha \in R\},$$

then all zeros of $f \in M_R$ on the line segment

$$R_2 := \{z \in R \mid \Im(z) < 2\}$$

other than the zeros at

$$i, \quad i\sqrt{2} \quad \text{and} \quad i\sqrt{3}$$

are transcendental.

Remark 2.6. Let g_k be any family of WH modular forms having all their zeros on the arc A . See Remark 2.3 for examples of such families. For $z \in \mathfrak{H}$, let

$$f_k^L(z) := g_k(\gamma_L^{-1}z) \quad \text{and} \quad f_k^R(z) := g_k(\gamma_R^{-1}z),$$

where $\gamma_L = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix}$ and $\gamma_R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then f_k^L (resp. f_k^R) have all their zeros on L (resp. on R).

Finally we discuss the values of WH modular forms. This can be done by suitably modifying the arguments followed in the work of the first author, Murty and Rath [10]. For the sake of completion, we indicate the relevant modifications and sketch the proofs. In order to do so, let us define an equivalence relation on $M^1(F)$, the graded ring (graded by the weight k) of WH modular forms with Fourier coefficients in a subfield F of $\overline{\mathbb{Q}}$ as follows. Two such WH modular forms f and g are called equivalent, denoted by $f \sim g$, if there exists natural numbers k_1, k_2 such that $f^{k_2} = \lambda g^{k_1}$ with $\lambda \in F^*$. Otherwise we call f not equivalent to g and write $f \not\sim g$. The restrictions of this relation to the graded rings of modular forms and cusp forms with Fourier coefficients in F give rise to equivalence relations on those rings.

Theorem 9. Let f be any non-zero element of $M_k^1(\overline{\mathbb{Q}})$. Suppose that $f \not\sim \Delta$ and $\alpha \in \mathfrak{H}$ is algebraic. Then $f^{12}(\alpha)/\Delta^k(\alpha)$ is algebraic if and only if α is a CM point.

Remark 2.7. It was noticed in [10] that if α is a CM point, then $\Delta(\alpha)$ is transcendental by a theorem of Schneider (see [22]). Further it was shown that if α is transcendental with $j(\alpha) \in \overline{\mathbb{Q}}$, then also $\Delta(\alpha)$ is transcendental. Moreover, if $\alpha \in \mathfrak{H}$ is non-CM algebraic, a conjecture of Nesterenko (see [18], page 31) will imply the transcendence of $\Delta(\alpha)$. Using the Open Mapping Theorem, we see that $\Delta(\alpha)$ can take algebraic values. It is then clear from the above discussion that in this case α is conjecturally transcendental and $j(\alpha)$ is transcendental.

Let $\alpha \in \mathfrak{H}$. Now if $\alpha \in \overline{\mathbb{Q}}$ or $j(\alpha) \in \overline{\mathbb{Q}}$, we can deduce the nature of $f(\alpha)$ except in one case. More precisely, we have the following theorems.

Theorem 10. *Let $f \in M_k^1(\overline{\mathbb{Q}})$ be non-zero and $\alpha \in \mathfrak{H}$ be such that $j(\alpha) \in \overline{\mathbb{Q}}$. Then either $f(\alpha) = 0$ or $f(\alpha)$ is algebraically independent with $e^{2\pi i \alpha}$.*

Theorem 11. *Let $\alpha \in \mathfrak{H}$ be a non-CM algebraic number. Also, let*

$$S_\alpha := \left\{ f \in M^1(\overline{\mathbb{Q}}) \mid f \neq 0 \text{ and } f(\alpha) \in \overline{\mathbb{Q}} \right\} / \sim.$$

Then S_α has at most one element.

Remark 2.8. We note that a conjecture of Nesterenko (discussed in Section 3) will imply that S_α is empty as was noticed earlier in [10] in the case of modular forms.

Finally, we have the following general theorem about the nature of zeros of WH modular forms.

Theorem 12. *Let $f \in M_k^1(\overline{\mathbb{Q}})$ be non-zero. Any zero of f is either CM or transcendental.*

3. PRELIMINARIES

We begin by fixing some notations and recalling some results. As before, an element $\alpha \in \mathfrak{H}$ lying in an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$ will be called a CM point. It is known, from classical theory of complex multiplication that if $\alpha \in \mathfrak{H}$ is a CM point, then $j(\alpha)$ is an algebraic integer, lying in the Hilbert class field of $\overline{\mathbb{Q}(\alpha)}$, the Galois closure of $\mathbb{Q}(\alpha)$. See Chapter 3 of [5] and Chapter 10 of [15] for further details on the theory of complex multiplication. Moreover, from Theorem 5 of [15], one can get the following proposition.

Proposition 13. *Let $\alpha_0 \in \mathfrak{H}$ be a CM point and also let $|D|$ be the absolute value of the discriminant of the lattice $\Lambda := (\alpha_0, 1)$. Then there is an element $\sigma \in \text{Gal}(\overline{\mathbb{Q}(\sqrt{D})}/\mathbb{Q}(\sqrt{D}))$ such that $\sigma(j(\alpha_0)) = j(\alpha_1)$, where*

$$\alpha_1 := \begin{cases} \frac{i\sqrt{|D|}}{2} & \text{if } D \equiv 0 \pmod{4}, \\ \frac{-1+i\sqrt{|D|}}{2} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

On the other hand, for algebraic points in the upper half plane, Schneider [22] proved the following result.

Theorem 14. (Schneider) *For $\alpha \in \mathfrak{H}$, if α and $j(\alpha)$ are algebraic, then α is CM.*

The above theorem of Schneider along with the following result of Nesterenko [17] play an important role in order to understand the nature of values of modular forms as shown in [10].

Theorem 15. (Nesterenko) *Let $\alpha \in \mathfrak{H}$. Then at least three of the four numbers*

$$e^{2\pi i\alpha}, \quad E_2(\alpha), \quad E_4(\alpha), \quad E_6(\alpha)$$

are algebraically independent.

Here for $z \in \mathfrak{H}$, we have

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n.$$

It is a holomorphic function on \mathfrak{H} and almost transforms like a modular form.

Furthermore, Nesterenko ([18], page 31) suggests the following general conjecture which generalizes both his and Schneider's theorem.

Conjecture 16. (Nesterenko) *Let $\alpha \in \mathfrak{H}$ and assume that at most three of the following five numbers*

$$\alpha, \quad e^{2\pi i\alpha}, \quad E_2(\alpha), \quad E_4(\alpha), \quad E_6(\alpha)$$

are algebraically independent. Then α is necessarily a CM point and the field

$$\overline{\mathbb{Q}}(e^{2\pi i\alpha}, E_2(\alpha), E_4(\alpha), E_6(\alpha))$$

has transcendence degree 3.

The following theorem by the first author, Murty and Rath (see [10]) will play a significant role in proving Theorem 10.

Theorem 17. *Let $\alpha \in \mathfrak{H}$ be such that $j(\alpha) \in \overline{\mathbb{Q}}$. Then $e^{2\pi i\alpha}$ and $\Delta(\alpha)$ are algebraically independent.*

4. PROOFS OF THEOREMS

From now on, every WH modular form is assumed to be non-zero and with algebraic Fourier coefficients unless otherwise stated.

For any subfield F of $\overline{\mathbb{Q}}$ and any WH modular form $f \in M_k^!(F)$, we define an associated function g_f given by

$$g_f(z) := \frac{f^{12}(z)}{\Delta^k(z)}, \quad \text{where } z \in \mathfrak{H}.$$

Clearly g_f is a WH modular form of weight 0 and hence a rational function in j . Since Δ does not vanish on \mathfrak{H} , it follows that g_f is a polynomial in j with coefficients in F . We will denote this polynomial by P_f .

4.1. Proofs of Theorem 9 and Theorem 12. Let $f \in M_k^!(\overline{\mathbb{Q}})$ be a non-zero WH modular form such that $f \not\sim \Delta$. Hence P_f is a non-constant polynomial with algebraic coefficients. Now if $g_f(\alpha)$ is algebraic, we have $j(\alpha)$ is algebraic. Moreover, if α is algebraic, we have by Schneider's theorem that α is CM. Conversely, if α is CM, then $j(\alpha)$ is algebraic and hence $g_f(\alpha)$ is algebraic. This completes the proof of Theorem 9.

If α is an algebraic zero of f , from the previous argument it follows that $j(\alpha) \in \overline{\mathbb{Q}}$ and hence by using Schneider's theorem, we get Theorem 12.

4.2. Proofs of Theorem 10 and Theorem 11. Let $\alpha \in \mathfrak{H}$ be such that $j(\alpha) \in \overline{\mathbb{Q}}$ and f be as in Theorem 10. Since $j(\alpha)$ is algebraic, by Theorem 17, we know that $e^{2\pi i \alpha}$ and $\Delta(\alpha)$ are algebraically independent. Now suppose that $f(\alpha)$ is not equal to zero. Since the non-zero number $g_f(\alpha)$ is a polynomial in $j(\alpha)$ with algebraic coefficients, it is algebraic. Hence $e^{2\pi i \alpha}$ and $f(\alpha)$ are algebraically independent. This completes the proof of Theorem 10.

We now prove Theorem 11. Let $\alpha \in \mathfrak{H}$ be a non-CM algebraic number and $f_1, f_2 \in S_\alpha$ be WH modular forms of weight k_1 and k_2 respectively. By Theorem 12, neither $f_1(\alpha)$ nor $f_2(\alpha)$ is equal to zero. For $z \in \mathfrak{H}$, consider the WH modular form

$$F(z) := f_1^{k_2}(\alpha) f_2^{k_1}(z) - f_2^{k_1}(\alpha) f_1^{k_2}(z)$$

of weight $k_1 k_2$. Again by Theorem 12, any zero of this WH modular form is either CM or transcendental. Since α is a non-CM algebraic number, we get a contradiction unless F is identically zero. Hence $f_1 \sim f_2$. This completes the proof.

4.3. Proofs of Theorem 1 and Corollary 2. Let f be as in Theorem 1. Without loss of generality, we can assume that $a(n_0) = 1$. Hence P_f is a monic polynomial with algebraic coefficients. Note that for any $\alpha \in \mathfrak{H}$,

$$f(\alpha) = 0 \iff P_f(j(\alpha)) = 0.$$

We will now prove Theorem 1 by method of contradiction. Suppose that all zeros of f are algebraic. Then by Theorem 12 they are CM points. Further, we know from the theory of complex multiplication that $j(\alpha)$ is an algebraic integer when α is CM. Since not all Fourier coefficients of f are algebraic integers, the same is true for the polynomial P_f . But this can not be true if all zeros of P_f are algebraic integers. This completes the proof of Theorem 1.

Let f be as in Corollary 2. Now if $f \in M_k^!(\overline{\mathbb{Q}})$ and $f \notin M_k^!(\overline{\mathbb{Z}})$, then by Theorem 1, f has at least one transcendental zero, a contradiction. This completes the proof of Corollary 2.

4.4. Proof of Theorem 3. Let $f \in M_k^!(\mathbb{Q})$ be a non-zero WH modular form with all its zeros on the arc A . Then P_f is a non-zero polynomial with rational coefficients. Let $\alpha_0 \in \mathfrak{H}$ be an algebraic zero of f . Then $j(\alpha_0)$ is algebraic and hence α_0 is a CM point. Let $|D|$ be the absolute value of the discriminant of the lattice $\Lambda := (\alpha_0, 1)$. Then by Proposition 13, there is an element $\sigma \in \text{Gal}(\overline{\mathbb{Q}(\sqrt{D})}/\mathbb{Q}(\sqrt{D}))$ such that $\sigma(j(\alpha_0)) = j(\alpha_1)$, where

$$\alpha_1 := \begin{cases} \frac{i\sqrt{|D|}}{2} & \text{if } D \equiv 0 \pmod{4}, \\ \frac{-1+i\sqrt{|D|}}{2} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Since $P_f(x) \in \mathbb{Q}[x]$, applying σ to $P_f(j(\alpha_0)) = 0$, we have $P_f(j(\alpha_1)) = 0$. This implies that $f(\alpha_1) = 0$. By assumption, all zeros of f are on the arc A . Hence $|\alpha_1| = 1$ and $-1/2 \leq \Re(\alpha_1) \leq 0$. Hence the only possible values of D are $D = -3, -4$, giving $\alpha_0 = \alpha_1 = i, \rho$. This completes the proof of Theorem 3.

4.5. Proof of Theorem 4. Let

$$g := \prod_{\gamma \in \Gamma/\Gamma_0(p)} f|_{\gamma}.$$

Then g is a WH modular form of level one with rational Fourier coefficients. This is true because of the q -expansion principle due to Deligne and Rapoport [6] (Theorem 3.9, p. 304) which tells us that if an integer weight modular form f has rational Fourier coefficients at the cusp $i\infty$, then the Fourier expansion of f at all other cusps must also have rational Fourier coefficients. Since all zeros of f lie on B , it follows that all zeros of g are on arc A . Hence by Theorem 3, we have that all these zeros are transcendental other than i and ρ . As a coset decomposition of Γ in $\Gamma_0(p)$, we may choose the $p+1$ elements (see page 8 of [13]) ST^n ($0 \leq n < p$) and I , where

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus all zeros of f are transcendental except the possible CM zeros in C .

4.6. Proofs of Theorem 5 and Theorem 6. Let $p = 2, 3$. By the given hypothesis, $E_{k,p}(z)$ is a modular form for $\Gamma_0(p)$. Consider the level one and weight k_p modular form

$$f_p(z) := \prod_{\gamma \in \Gamma/\Gamma_0(p)} E_{k,p}(z)|_{\gamma}.$$

By the Deligne-Rapoport q -expansion principle, the modular form f_p has rational Fourier coefficients. Then P_{f_p} is a non-constant polynomial with rational coefficients. Let α_p be a CM zero of g_{f_p} . Then by Proposition 13, there exists an element $\sigma \in \text{Gal}(\overline{\mathbb{Q}(\sqrt{D})}/\mathbb{Q}(\sqrt{D}))$ such that

$\sigma(j(\alpha_p)) = j(\alpha'_p)$, where

$$\alpha'_p := \begin{cases} \frac{i\sqrt{|D|}}{2} & \text{if } D \equiv 0 \pmod{4}, \\ \frac{-1+i\sqrt{|D|}}{2} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Since $P_{f_p}(x) \in \mathbb{Q}[x]$, applying σ to $P_{f_p}(j(\alpha_p)) = 0$, we have $P_{f_p}(j(\alpha'_p)) = 0$. This implies that $f_p(\alpha'_p) = 0$. But then either $E_{k,p}(\alpha'_p) = 0$ or $E_{k,p}(\gamma\alpha'_p) = 0$, where $\gamma \in \{I, S, ST\}$ if $p = 2$ and $\gamma \in \{I, S, ST, ST^2\}$ if $p = 3$. Here S, T are as in proof of Theorem 4. By the work of Miezaki, Nozaki and Shigezumi (see [16]), we know that all zeros of $E_{k,p}(z)$ for $p = 2, 3$ lie on the arc

$$A_p := \{z \in \mathfrak{H} \mid -1/2 \leq \Re(z) \leq 0, |z| = 1/\sqrt{p}\}.$$

Hence the only possible CM zeros of $E_{k,2}(z)$ in \mathfrak{F}_2 can be calculated to be

$$\frac{i}{\sqrt{2}}, \quad \frac{-1+i\sqrt{7}}{4}, \quad \frac{-1+i}{2}$$

and the only possible CM zeros of $E_{k,3}(z)$ in \mathfrak{F}_3 can be calculated to be

$$\frac{i}{\sqrt{3}}, \quad \frac{-1+i\sqrt{11}}{6}, \quad \frac{-1+i\sqrt{2}}{3}, \quad \frac{-3+i\sqrt{3}}{6}.$$

4.7. Proofs of Theorem 7 and Theorem 8. Let $J = L$ or R and $f_J \in M_k^!(\mathbb{Q})$ be a non-zero WH modular form with all its zeros on the half-line J . Let $\alpha_{0,J} \in \mathfrak{H}$ be a CM zero of f_J and $|D_J|$ be the absolute value of the discriminant of the lattice $\Lambda := (\alpha_{0,J}, 1)$. Then arguing as before and using Proposition 13, we see that there is a $\sigma_J \in \text{Gal}(\overline{\mathbb{Q}(\sqrt{D_J})}/\mathbb{Q}(\sqrt{D_J}))$ such that $\sigma(j(\alpha_{0,J})) = j(\alpha_{1,J})$, where

$$\alpha_{1,J} := \begin{cases} \frac{i\sqrt{|D_J|}}{2} & \text{if } D_J \equiv 0 \pmod{4}, \\ \frac{-1+i\sqrt{|D_J|}}{2} & \text{if } D_J \equiv 1 \pmod{4}. \end{cases}$$

It is easy to see that $\alpha_{1,J}$ is a zero of f_J . By assumption, all zeros of f_J lie on J . Hence

$$\Re(\alpha_{1,J}) = \begin{cases} -\frac{1}{2} & \text{if } J = L, \\ 0 & \text{if } J = R \end{cases}$$

and we have

$$\alpha_{1,J} = \begin{cases} \frac{-1+i\sqrt{|D_L|}}{2} & \text{if } J = L, \\ \frac{i\sqrt{|D_R|}}{2} & \text{if } J = R, \end{cases}$$

i.e. $D_L \equiv 1 \pmod{4}$ and $D_R \equiv 0 \pmod{4}$. Since $\alpha_{0,J}$ is a CM point, it is a root of a polynomial $a_J x^2 + b_J x + c_J \in \mathbb{Z}[x]$ with $(a_J, b_J, c_J) = 1, a_J > 0$ and $b_J^2 - 4a_J c_J < 0$. Further, $D_J = b_J^2 - 4a_J c_J$

and $\alpha_{0,J} = \frac{-b_J + i\sqrt{|D_J|}}{2a_J}$. By assumption, we have

$$\Re(\alpha_{0,J}) = \begin{cases} -\frac{1}{2} & \text{if } J = L, \\ 0 & \text{if } J = R \end{cases}$$

and

$$\Im(\alpha_{0,L}) \geq \begin{cases} \sqrt{3}/2 & \text{if } J = L, \\ 1 & \text{if } J = R. \end{cases}$$

Hence

$$b_L = a_L \text{ and } 1 \leq a_L \leq \left\lceil \sqrt{\frac{|D_L|}{3}} \right\rceil$$

and

$$b_R = 0 \text{ and } 1 \leq a_R \leq \left\lceil \frac{\sqrt{|D_R|}}{2} \right\rceil.$$

This completes the proofs of the theorems.

4.8. Proofs of Remark 2.4 and Remark 2.5. Let $J = L$ or R and $f_J \in M_J$. Suppose that f_J has a CM zero on the line segment J_2 , where $J_2 = L_2$ or R_2 .

When $J_2 = L_2$, by Theorem 7, zeros of f_J are in the imaginary quadratic fields K with discriminant $D = -3, -7, -11, -15$. Next, we note that it is possible to have $a = 2$ in Theorem 7 only when $D = -15$. This completes the proof of the Remark 2.4.

Now if $J_2 = R_2$, then by Theorem 8, we have $D = -4, -8, -12$. For all these D 's we have $a = 1$. Hence the possible CM zeros of f_J on R_2 are i , $i\sqrt{2}$ and $i\sqrt{3}$. This completes the proof of the Remark 2.5.

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